JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON TUESDAY 28th JANUARY 2025) **TIME : 9:00 AM TO 12:00 NOON**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The number of different 5 digit numbers greater than 50000 that can be formed using the digits 0, 1, 2, 3, 4, 5, 6, 7, such that the sum of their first and last digits should not be more than 8, is

(1) 4608 (2) 5720

(3) 5719 (4) 4607

Ans. (4)

Sol. Case I 5 _ _ _ 0

Case II 5 _ _ _ 1

5 2

5 3

6 0

6 1

6 2

7 0

7 _ _ _ 1

$9\times(8\times8\times8) = 4608$ but 50000 is not included, so total numbers $4608 - 1 = 4607$

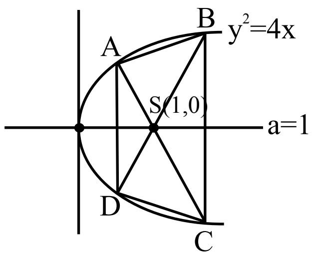
2. Let ABCD be a trapezium whose vertices lie on the parabola $y^2 = 4x$. Let the sides AD and BC of the trapezium be parallel to y-axis. If the diagonal AC is of length $\frac{25}{4}$ and it passes through the point (1, 0), then the area of ABCD is :

(1) $\frac{75}{4}$ (2) $\frac{25}{2}$

(3) $\frac{125}{8}$ (4) $\frac{75}{8}$

Ans. (1)

Sol.



$A(at_1^2, 2at_1)$ & $C\left(\frac{a}{t_1^2}, -\frac{2a}{t_1}\right)$

Length $AC = a\left(t_1 + \frac{1}{t_1}\right)^2 = \frac{25}{4}$, $t_1 + \frac{1}{t_1} = \pm\frac{5}{2}$

$\Rightarrow t_1 = 2$ or $\frac{1}{2}$, $A\left(\frac{1}{2}, 1\right)$, $D\left(\frac{1}{4}, -1\right)$, $B(4, 4)$, $C(4, -4)$

So, area of trapezium $= \frac{1}{2}(8+2)\left(4 - \frac{1}{4}\right) = \frac{75}{4}$

3. Two number k_1 and k_2 are randomly chosen from the set of natural numbers. Then, the probability that the value of $i^{k_1} + i^{k_2}$, $(i = \sqrt{-1})$ is non-zero, equals

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$

(3) $\frac{3}{4}$ (4) $\frac{2}{3}$

Ans. (3)

Sol. $i^{k_1} + i^{k_2} \neq 0$ $\quad i^{k_1} \rightarrow 4$ option for i, –1, –i, 1

Total cases $\Rightarrow 4\times4 = 16$

Unfovourble cases $\Rightarrow i^{k_1} + i^{k_2} = 0$

$$\left\{\begin{matrix} 1, -1 \\ -1, 1 \\ i, -i \\ -i, i \end{matrix}\right\}$$

4 Cases $\Rightarrow$ Probability $= \frac{16-4}{16} = \frac{3}{4}$

4. If $f(x) = \frac{2^x}{2^x + \sqrt{2}}, x \in R$, then $\sum_{k=1}^{81} f\left(\frac{k}{82}\right)$ is equal to :

(1) 41 (2) $\frac{81}{2}$

(3) 82 (4) $81\sqrt{2}$

Ans. (2)

Sol. $f(x)=\dfrac{2^x}{2^x+\sqrt{2}}$

$f(x)+f(1-x)=\dfrac{2^x}{2^x+\sqrt{2}}+\dfrac{2^{1-x}}{2^{1-x}+\sqrt{2}}$

$=\dfrac{2^x}{2^x+\sqrt{2}}+\dfrac{2}{2+\sqrt{2}\,2^x}=\dfrac{2^x+\sqrt{2}}{2^x+\sqrt{2}}=1$

Now, $\sum_{k=1}^{81} f\left(\dfrac{k}{82}\right)=f\left(\dfrac{1}{82}\right)+f\left(\dfrac{2}{82}\right)+.......+f\left(\dfrac{81}{82}\right)$

$=f\left(\dfrac{1}{82}\right)+f\left(\dfrac{1}{82}\right)+.......+f\left(1-\dfrac{2}{82}\right)+f\left(1-\dfrac{1}{82}\right)$

$\left[f\left(\dfrac{1}{82}\right)+f\left(1-\dfrac{1}{82}\right)\right]+\left[f\left(\dfrac{2}{82}\right)+f\left(1-\dfrac{2}{82}\right)\right]+.....40\text{ cases}+f\left(\dfrac{41}{82}\right)$

$(1+1+.....+1)\,40\text{ times}+\dfrac{2^{1/2}}{2^{1/2}+2^{1/2}}$

$40+\dfrac{1}{2}=\dfrac{81}{2}$

5. Let $f: R \rightarrow R$ be a function defined by $f(x) = (2 + 3a)x^2 + \left(\dfrac{a+2}{a-1}\right)x + b,\ a \neq 1$. If

$f(x + y) = f(x) + f(y) + 1 - \dfrac{2}{7}xy$, then the value of

$28\sum_{i=1}^{5}|f(i)|$ is:

(1) 715 (2) 735

(3) 545 (4) 675

Ans. **(4)**

Sol. $f(x)=(3a+2)x^2+\left(\dfrac{a+2}{a-1}\right)x+b$

$f\left(x+\dfrac{1}{2}\right)=f(x)+f(y)+1-\dfrac{2}{7}xy$ ......(1)

In (1) Put $x = y = 0 \Rightarrow f(0) = 2f(0) + 1 \Rightarrow f(0) = -1$

So, $f(0) = 0 + 0 + b = -1 \Rightarrow b = -1$

In (1) Put $y = -x \Rightarrow f(0) = f(x) + f(-x) + 1 + \dfrac{2}{7}x^2$

$-1 = 2(3a + 2)x^2 + 2b + 1 + \dfrac{2}{7}x^2$

$-1=\left(2(3a+2)+\dfrac{2}{7}\right)x^2+1-2$

$\Rightarrow 6a + 4 + \dfrac{2}{7} = 0$

$a=-\dfrac{5}{7}$

So $f(x)=-\dfrac{1}{7}x^2-\dfrac{3}{4}x-1$

$\Rightarrow |f(x)|=\dfrac{1}{28}\left|4x^2+21x+28\right|$

Now, $28\sum_{i=1}^{5}|f(6)|=28\left(|f(1)|+f(2)+...+f(5)\right)$

$28.\dfrac{1}{28}.675=675$

6. Let A(x, y, z) be a point in xy-plane, which is equidistant from three points (0, 3, 2), (2, 0, 3) and (0, 0, 1).

Let B = (1, 4, –1) and C = (2, 0, –2). Then among the statements

(S1) : ΔABC is an isosceles right angled triangle and

(S2) : the area of ΔABC is $\dfrac{9\sqrt{2}}{2}$.

(1) both are true (2) only (S1) is true

(3) only (S2) is true (4) both are false

Ans. **(2)**

Sol. A(x,y,z) Let P (0,3,2), Q(2,0,3), R(0,0,1)

AP = AQ = AR

$x^2 + (y-3)^2 + (z-2)^2 = (x-2)^2 + y^2 + (z-3)^2 = x^2 + y^2 + (z-1)^2$

In xy plane z = 0

So, $x^2 - 4x + 4 + y^2 + 9 = x^2 + y^2 + 1$

x = 3

$9 + y^2 - 6y + 9 + 4 = x^2 + y^2 + 1$

So, A(3,2,0) also B(1,4,–1) & C(2,0,–2)

Now AB = $\sqrt{4+4+1}=3$

AC = $\sqrt{1+4+4}=3$

BC $=\sqrt{1+16+1}=\sqrt{18}$

AB = AC

isosceles Δ & $AB^2 + AC^2 = BC^2$

right angle Δ

Area of ΔABC = $\frac{1}{2}$ ×base.height

$\frac{1}{2}\times 3\times 3=\frac{9}{2}$

So only S_1 is true

7. The relation R={(x, y) : x, y ∈ z and x + y is even} is :

(1) reflexive and transitive but not symmetric

(2) reflexive and symmetric but not transitive

(3) an equivalence relation

(4) symmetric and transitive but not reflexive

Ans. (3)

Sol. R = {(x,y), x + y is even x, y ∈ z}

reflexive x + x = 2x even

symmetric of x + y is even, then (y + x) is also even

transitive of x + y is even & y + z is even then x + z is also even

So, relation is an equivalence relation.

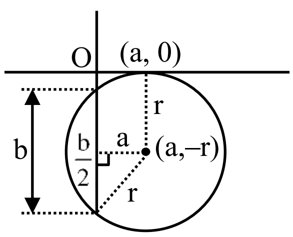
8. Let the equation of the circle, which touches x-axis at the point (a, 0), a > 0 and cuts off an intercept of length b on y-axis be $x^2 + y^2 - \alpha x + \beta y + \gamma = 0$. If the circle lies below x-axis, then the ordered pair $(2a, b^2)$ is equal to :

(1) $(\alpha, \beta^2 + 4\gamma)$ (2) $(\gamma, \beta^2 - 4\alpha)$

(3) $(\gamma, \beta^2 + 4\alpha)$ (4) $(\alpha, \beta^2 - 4\gamma)$

Ans. (4)

Sol.



By pytogorus $r^2 = a^2 + \frac{b^2}{4} = P^2$

$r=\sqrt{\frac{4a^2+b^2}{4}}$

Equation of circle is $(x-\alpha)^2 + (y-\beta)^2 = r^2$

$x^2 + y^2 - 2ax - 2py + \alpha^2 + p^2 - r^2 = 0$

comparision $x^2 + y^2 - \alpha x + \beta y + r = 0$

$-\alpha = -2a,\ \beta = -2p,\ r = a^2$

$\Rightarrow 2a = \alpha,\ \ 4a^2 + b^2 = 4p^2$

$\alpha^2 + b^2 = 4p^2$

$\alpha^2 + b^2 = \beta^2$

So, $(2a, b^2) = (\alpha, \beta^2 - 4r)$

9. Let $<a_n>$ be a sequence such that $a_0 = 0$, $a_1 = \frac{1}{2}$ and $2a_{n+2} = 5a_{n+1} - 3a_n$, n = 0, 1, 2, 3, …… Then $\sum_{k=1}^{100} a_k$ is equal to :

(1) $3a_{99} - 100$ (2) $3a_{100} - 100$

(3) $3a_{100} + 100$ (4) $3a_{99} + 100$

Ans. (2)

Sol. $a_0 = 0,\ a_1 = \frac{1}{2}$

$2a_{n+2} = 5a_{n+1} - 3a_n$

$2x^2 - 5x + 3 = 0 \Rightarrow x = 1, 3/2$

$\therefore\ a_n = A1^n + B\left(\frac{3}{2}\right)^n$

$$\left.\begin{array}{ll} n=0 & 0=A+B \\ n=1 & \frac{1}{2}=A+\frac{3}{2}B \end{array}\right] \begin{array}{l} A=-1 \\ B=1 \end{array}$$

$\Rightarrow a_n = -1 + \left(\frac{3}{2}\right)^n$

$$\sum_{k=1}^{100} a_k = \sum_{k=1}^{100}(-1)+\left(\frac{3}{2}\right)^k$$

$$=-100+\frac{\left(\frac{3}{2}\right)\left(\left(\frac{3}{2}\right)^{100}-1\right)}{\frac{3}{2}-1}$$

$$=-100+3\left(\left(\frac{3}{2}\right)^{100}-1\right)$$

$= 3.(a_{100}) - 100$

10. $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}\right)$ is equal to :

(1) 1 (2) 0

(3) $\frac{33}{65}$ (4) $\frac{32}{65}$

Ans. (2)

Sol. $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{33}{65}\right)$

$\cos\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{33}{56}\right)$

$\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{5}{12}}{1+\frac{3}{4}\cdot\frac{5}{12}}\right)+\tan^{-1}\frac{33}{56}\right)$

$\cos\left(\tan^{-1}\frac{56}{33}+\cot^{-1}\frac{56}{33}\right)$

$\cos\left(\frac{\pi}{2}\right)=0$

11. Let T_r be the r^{th} term of an A.P. If for some m, $T_m = \frac{1}{25}$, $T_{25} = \frac{1}{20}$ and $20\sum_{r=1}^{25}T_r = 13$, then $5m\sum_{r=m}^{2m}T_r$ is equal to :

(1) 112 (2) 126

(3) 98 (4) 142

Ans. (2)

Sol. $T_m = \frac{1}{25}, T_{25} = \frac{1}{20}, 20\sum_{r=1}^{25}T_r = 13$

$T_m = a + (m-1)d = \frac{1}{25}$ ......(1)

$T_{25} = a + 24d = \frac{1}{20}$

$20.\frac{25}{2}\left[a+\frac{1}{20}\right]=13 \Rightarrow a=\frac{1}{500}$

also, $20S_{25} = 20.\frac{25}{2}[2a+24d]=13 \Rightarrow d=\frac{1}{500}$

from (1) $\frac{1}{500}+\frac{m-1}{500}=\frac{1}{25} \Rightarrow m=20$

Now,

$5m\sum_{r=m}^{2m}T_r = 100\sum_{r=20}^{40}T_r = 126$

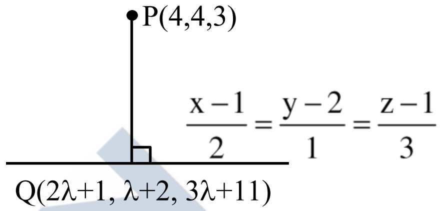
12. If the image of the point (4, 4, 3) in the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-1}{3}$ is (α, β, γ), then $\alpha+\beta+\gamma$ is equal to

(1) 9 (2) 12

(3) 8 (4) 7

Ans. (1)

Sol.



$\overrightarrow{P\alpha} \perp \left(2\hat{i}+\hat{j}+3\hat{k}\right)$

$\Rightarrow 2(2\lambda-3)+1(\lambda-2)+3(3\lambda-2)=0$

$\Rightarrow 14\lambda-14=0, \lambda=1$

So, Q(3,3,4)

Let image in $R(\alpha,\beta,\gamma)$

$\frac{\alpha+\gamma}{2}=3, \frac{\beta+\gamma}{2}=3, \frac{\gamma+3}{2}=4$

$(\alpha, \beta, \gamma) = (2, 2, 5)$

$\Rightarrow \alpha+\beta+\gamma = 9$

13. If $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{96x^2\cos^2 x}{(1+e^x)}dx = \pi(\alpha\pi^2+\beta)$, $\alpha, \beta \in Z$, then $(\alpha+\beta)^2$ equals :

(1) 144 (2) 196

(3) 100 (4) 64

Ans. (3)

Sol. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{96x^2\cos^2 x}{(1+e^x)}dx$ (Apply King Property)

$\int_0^{\frac{\pi}{2}}96x^2\cos^2 x = 48\int_0^{\frac{\pi}{2}}x^2(1+\cos 2x)dx$

$48\left[\left(\frac{x^3}{3}\right)_0^{\pi/2}+\int_0^{\frac{\pi}{2}}\underset{I}{x^2}\underset{II}{\cos 2x}\,dx\right]$

$\Rightarrow$ On solving $\pi(2\pi^2-12)$

$\Rightarrow \alpha = 2, \beta = -12$

$\Rightarrow (\alpha+\beta)^2 = 100$

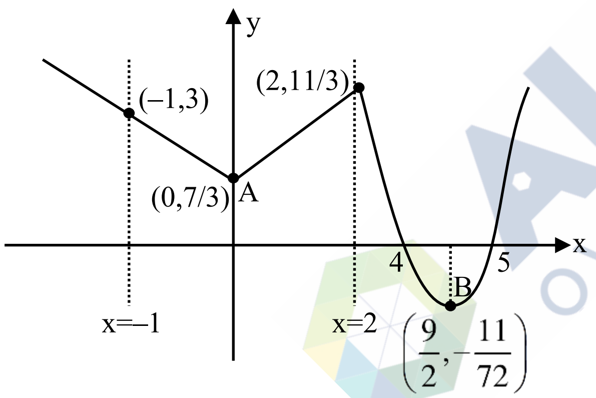
14. The sum of all local minimum values of the function $f(x)=\begin{cases}1-2x, & x<-1\\ \frac{1}{3}(7+2|x|), & -1\le x\le 2\\ \frac{11}{18}(x-4)(x-5), & x>2\end{cases}$

(1) $\frac{171}{72}$ (2) $\frac{131}{72}$

(3) $\frac{157}{72}$ (4) $\frac{167}{72}$

Ans. **(3)**

Sol. $f(x)=\begin{cases}1-2x, & x<-1\\ \frac{1}{3}(7-2x), & -1\le x\le 2\\ \frac{1}{3}(7+2x) & 0\le x<2\\ \frac{11}{18}(x-4)(x-5), & x>2\end{cases}$



$\therefore$ Local minimum values at A & B

$\frac{7}{3}-\frac{11}{72}$

$\Rightarrow \frac{168-11}{72} \Rightarrow \frac{157}{72}$

15. The sum, of the squares of all the roots of the equation $x^2+|2x-3|-4=0$, is :

(1) $3(3-\sqrt{2})$ (2) $6(3-\sqrt{2})$

(3) $6(2-\sqrt{2})$ (4) $3(2-\sqrt{2})$

Ans. **(3)**

Sol. $x^2+|2x-3|-4=0$

Case I : $x\ge\frac{3}{2}$

$x^2+2x-3-4=0$

$x^2+2x-7=0$

$x=2\sqrt{2}-1$

Case II : $x<\frac{3}{2}$

$x^2+3-2x-4=0$

$x^2-2x-1=0$

$x=1-\sqrt{2}$

Sum of squares $=\left(2\sqrt{2}-1\right)^2+\left(1-\sqrt{2}\right)^2$

$=8-4\sqrt{2}+1+1-2\sqrt{2}+2$

$=6\left(2-\sqrt{2}\right)$ $\therefore$ (3)

16. Let for some function $y=f(x)$, $\int_0^x t\,f(t)dt=x^2f(x)$, $x>0$ and $f(2)=3$. Then $f(6)$ is equal to :

(1) 1 (2) 2

(3) 6 (4) 3

Ans. **(1)**

Sol. $\int_0^x tf(t)dt=x^2+(x), x>0$

Diff. both side w.r. to x

$xf(x)=x^2f'(x)+2xf(x)$

$-xf(x)=x^2f'(x)$

$\int\frac{f'(x)}{f(x)}dx=\int\frac{-1}{2}dx$

logdf(x) = –logx + logc

$f(x)=\frac{c}{x}$

$f(2)=3\Rightarrow 3=\frac{c}{2}\Rightarrow c=6$

$f(x)=\frac{6}{x}$

$f(6)=1$ $\therefore$ (1)

17. Let ${}^nC_{r-1}=28$, ${}^nC_r=56$ and ${}^nC_{r+1}=70$. Let A(4cost, 4sint), B(2sint, –2cost) and C($3r-n$, r^2-n-1) be the vertices of a triangle ABC, where t is a parameter. If $(3x-1)^2+(3y)^2=\alpha$, is the locus of the centroid of triangle ABC, then α equals :

(1) 20 (2) 8

(3) 6 (4) 18

Ans. **(1)**

Sol. ${}^{n}C_{r-1} = 28,\ {}^{n}C_{r} = 56$

$$\frac{{}^{n}C_{r-1}}{{}^{n}C_{r}} = \frac{28}{56}$$

$$\frac{\dfrac{n!}{(r-1)!(n-r+1)!}}{\dfrac{n!}{r!(n-r)!}} = \frac{1}{2}$$

$$\frac{r}{(n-r+1)} = \frac{1}{2}$$

$3r = n+1$ —(i)

$$\frac{{}^{n}C_{r}}{{}^{n}C_{r+1}} = \frac{56}{70}$$

$$\frac{(r+1)}{(n-r)} = \frac{56}{70} \Rightarrow 9r = 4n-5 \quad \text{—(ii)}$$

By (i) & (ii)

$(r = 3), (n = 8)$

A $(4\cos t, 4\sin t)$ B$(2\sin t, -2\cos t)$ C$(3r-n, r^2-n-1)$

A $(4\cos t, 4\sin t)$ B$(2\sin t, -2\cos t)$ C$(1, 0)$

$(3x-1)^2 + (3y)^2 = (4\cos t + 2\sin t)^2 + (4\sin t - \cos t)^2$

$(3x-1)^2 + (3y)^2 = 20 \quad \therefore (1)$

18. Let O be the origin, the point A be $z_1 = \sqrt{3}+2\sqrt{2}i$, the point B($z_2$) be such that $\sqrt{3}|z_2| = |z_1|$ and $\arg(z_2) = \arg(z_1) + \frac{\pi}{6}$. Then

(1) area of triangle ABO is $\frac{11}{\sqrt{3}}$

(2) ABO is a scalene triangle

(3) area of triangle ABO is $\frac{11}{4}$

(4) ABO is an obtuse angled isosceles triangle

Ans. (4)

Sol. $z_1 = \sqrt{3}+2\sqrt{2}i$ & $\frac{|z_2|}{|z_1|} = \frac{1}{\sqrt{3}}$

given $\arg\left(\frac{z_2}{z_1}\right) = \frac{\pi}{6}$

$$z_2 = \frac{|z_2|}{|z_1|} \cdot z_1\, e^{i\left(\frac{\pi}{6}\right)}$$

$$z_2 = \frac{1}{\sqrt{3}} \cdot \frac{\left(\sqrt{3}+2\sqrt{2}i\right)\left(\sqrt{3}+i\right)}{2}$$

$$z_2 = \frac{\left(3-2\sqrt{2}\right)+i\left(2\sqrt{6}+\sqrt{3}\right)}{2\sqrt{3}}$$

Now,

$$z_1 - z_2 = \frac{\left(3+2\sqrt{2}\right)+i\left(2\sqrt{6}-\sqrt{3}\right)}{2\sqrt{3}}$$

$|z_1-z_2| = |z_2| \Rightarrow \Delta ABO$ is isosceles with angles $\frac{\pi}{6}, \frac{\pi}{6}$ & $\frac{2\pi}{3}$

19. Three defective oranges are accidently mixed with seven good ones and on looking at them, it is not possible to differentiate between them. Two oranges are drawn at random from the lot. If x denote the number of defective oranges, then the variance of x is :

(1) 28/75 (2) 14/25

(3) 26/75 (4) 18/25

Ans. (1)

Sol. 10 oranges → 7 good, 3 defected

Probability distribution

x_i	p_i
$x=0$	$\frac{7C_2}{10C_2} = \frac{42}{90}$
$x=1$	$\frac{7C_1 \times 3C_1}{10C_2} = \frac{42}{90}$
$x=2$	$\frac{3C_2}{10C_2} = \frac{6}{90}$

Now,

$$\mu = \sum x_i p_i = \frac{42}{90} + \frac{12}{90} = \frac{54}{90}$$

$$\sigma^2 = \sum p_i x_i^2 - \mu^2 = \frac{42}{90} + \frac{24}{90} - \left(\frac{54}{90}\right)^2$$

$$\Rightarrow \frac{66}{90} - \left(\frac{54}{90}\right)^2$$

$$\sigma^2 \Rightarrow \frac{28}{75} \quad \therefore (1)$$

20. The area (in sq. units) of the region

$\{(x, y): 0 \le y \le 2|x| + 1, 0 \le y \le x^2 + 1, |x| \le 3\}$

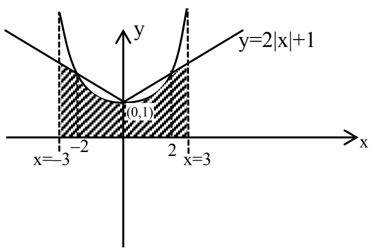
is

(1) $\frac{80}{3}$ (2) $\frac{64}{3}$

(3) $\frac{17}{3}$ (4) $\frac{32}{3}$

Ans. **(2)**

Sol.



$$\text{Area} = 2\left[\int_0^2 (x^2+1)dx + \int_2^3 (2x+1)dx\right]$$

$\Rightarrow \frac{64}{3}$ $\therefore$ (2)

SECTION-B

21. Let M denote the set of all real matrices of order 3 × 3 and let S = {–3, –2, –1, 1, 2}. Let

$S_1 = \{A = [a_{ij}] \in M : A = A^T \text{ and } a_{ij} \in S, \forall i, j\}$

$S_2 = \{A = [a_{ij}] \in M : A = -A^T \text{ and } a_{ij} \in S, \forall i, j\}$

$S_3 = \{A = [a_{ij}] \in M : a_{11} + a_{22} + a_{33} = 0 \text{ and } a_{ij} \in S, \forall i,j\}$

If $n(S_1 \cup S_2 \cup S_3) = 125\alpha$, then α equals.

Ans. **(1613)**

Sol. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

No. of elements in $S_1 : A = A^T \Rightarrow 5^3 \times 5^3$

No. of elements in $A = -A^T \Rightarrow 0$

since no. zero in 5

No. of elements in $S_3 \Rightarrow$

$$\left.\begin{array}{l} a_{11} + a_{22} + a_{33} = 0 \Rightarrow (1,2,-3) \Rightarrow 31 \\ \text{or} \\ (1,1,-2) \Rightarrow 3 \\ \text{or} \\ (-1,-1,2) \Rightarrow 3 \end{array}\right\} \Rightarrow 12 \times 5^6$$

$n(S_1 \cap S_3) = 12 \times 5^3$

$n(S_1 \cup S_2 \cup S_3) = 5^6(1+12) - 12 \times 5^3$

$\Rightarrow 5^3 \times [13 \times 5^3 - 12] = 125\alpha$

$\alpha = 1613$

22. If $\alpha = 1 + \sum_{r=1}^{6} (-3)^{r-1}\, {}^{12}C_{2r-1}$, then the distance of the point $(12, \sqrt{3})$ form the line $\alpha x - \sqrt{3}y + 1 = 0$ is ....

Ans. **(5)**

Sol. $\alpha = 1 + \sum_{r=1}^{6} (-1)^{r-1}\, {}^{12}C_{2r-1} 3^{r-1}$

$$\alpha = 1 + \sum_{r=1}^{6} {}^{12}C_{2r-1} \frac{(\sqrt{3}i)^{2t-1}}{\sqrt{3}i} \quad i = \text{iota}, \text{ let } \sqrt{3}i = x$$

$$\alpha = 1 + \frac{1}{\sqrt{3}i}\left({}^{12}C_1 x + {}^{12}C_3 x^3 + \ldots {}^{12}C_{11} x^{11}\right)$$

$$= 1 + \frac{1}{\sqrt{3}i}\left(\frac{(1+\sqrt{3}i)^{12} - (1-\sqrt{3}i)^{12}}{2}\right)$$

$$= 1 + \frac{1}{\sqrt{3}i}\left(\frac{(-2w^2)^{12} - (2w)^{12}}{2}\right) = 1$$

so distance of $(12, \sqrt{3})$ from $x - \sqrt{3}y + 1 = 0$ is

$\frac{12 - 3 + 1}{2} = 5$

23. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{d} = \vec{a} \times \vec{b}$. If $\vec{c}$ is a vector such that $\vec{a}.\vec{c} = |\vec{c}|$, $|\vec{c} - 2\vec{a}|^2 = 8$ and the angle between $\vec{d}$ and $\vec{c}$ is $\frac{\pi}{4}$, then $|10 - 3\vec{b}.\vec{c}| + |\vec{d} \times \vec{c}|^2$ is equal to .....

Ans. **(6)**

Sol. $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{d} = \vec{a} \times \vec{b}$

$= -\hat{i} + \hat{j}$

$|\vec{c} - 2\vec{a}|^2 = 8$

$|c|^2 + 4|a|^2 - 4(a.c) = 8$

$c^2 + 12 - 4c = 8$

$c^2 - 4c + 4 = 0$

$|c| = 2$

$\vec{d} = \vec{a} \times \vec{b}$

$\vec{d} \times \vec{c} = (\vec{a} \times \vec{b}) \times \vec{c}$

$\left(|a||c|\sin\frac{\pi}{4}\right)^2 = ((a.c).b - (b.c).a)^2$

$4 = 4b^2 + (b.c)2(a^2) - 2(b.c)(a.b)$

$4 = 36 + 3x^2 - 20x$

Let b.c = x

$3x^2 - 20x + 32 = 0$

$3x^2 - 12x - 8x + 32 = 0$

$x = \frac{8}{3}, 4$

$b.c = \frac{8}{3}, 4$

$b.c = \frac{8}{3}$

Now $|10 - 3b.c| + |d \times c|^2$

$|10 - 8| + (2)^2$

$\Rightarrow$ 6 Ans.

24. Let

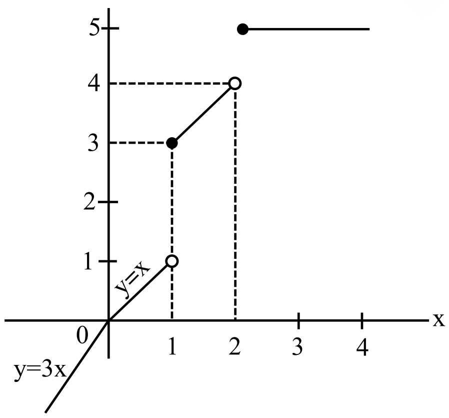
$$f(x) = \begin{cases} 3x, & x < 0 \\ \min\{1 + x + [x], x + 2[x]\}, & 0 \le x \le 2 \\ 5, & x > 2 \end{cases}$$

where [.] denotes greatest integer function. If α and β are the number of points, where f is not continuous and is not differentiable, respectively, then $\alpha + \beta$ equals…….

Ans. (5)

Sol. $$f(x) = \begin{cases} 3x & ; \quad x < 0 \\ \min\{1 + x, x\} & ; \quad 0 \le x < 1 \\ \min\{2 + x, x + 2\} & ; \quad 1 \le x < 2 \\ 5 & ; \quad x > 2 \end{cases}$$

$$f(x) = \begin{cases} 3x & ; \quad x < 0 \\ x & ; \quad 0 \le x < 1 \\ x + 2 & ; \quad 1 \le x < 2 \\ 5 & ; \quad x > 2 \end{cases}$$



Not continuous at $x \in \{1, 2\} \Rightarrow \alpha = 2$

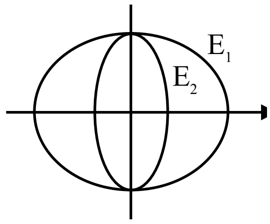
Not diff. at $x \in \{0, 1, 2\} \Rightarrow \beta = 3$

$\alpha + \beta = 5$

25. Let $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ be an ellipse. Ellipses E_i's are constructed such that their centres and eccentricities are same as that of E_1, and the length of minor axis of E_i is the length of major axis of E_{i+1} ($i \ge 1$). If A_i is the area of the ellipse E_i, then $\frac{5}{\pi}\left(\sum_{i=1}^{\infty} A_i\right)$, is equal to ……

Ans. (54)

Sol.



$E_1 = \frac{x^2}{9} + \frac{y^2}{4} \Rightarrow e = \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$

$E_2 : \frac{x^2}{a^2} + \frac{y^2}{4} = 1$

$e = \frac{\sqrt{5}}{3} = \sqrt{1 - \frac{a^2}{4}} \Rightarrow \frac{5}{9} = 1 - \frac{a^2}{4}$

$a^2 = \frac{16}{9}$

$E_2 : \frac{x^2}{\frac{16}{9}} + \frac{y^2}{4} = 1$

$E_3 : \frac{x^2}{\frac{16}{9}} + \frac{y^2}{b^2} = 1$

$e = \frac{\sqrt{5}}{3} = \sqrt{1 - \frac{b^2}{\frac{16}{9}}} \Rightarrow b^2 = \frac{64}{81}$

$E_3 = \frac{x^2}{\frac{16}{9}} + \frac{y^2}{\frac{64}{81}} = 1$

$A_1 = \pi \times 3 \times 2 \Rightarrow 6\pi$

$A_2 = \pi \times \frac{4}{3} \times 2 = \frac{8\pi}{3}$

$A_3 = \pi \times \frac{4}{3} \times \frac{8}{9} = \frac{32\pi}{81}$

$\sum_{i=1}^{\infty} A_i = 6\pi + \frac{8\pi}{3} + \frac{32\pi}{81} + \ldots\infty \Rightarrow \frac{6\pi}{1 - \frac{4}{9}} \Rightarrow \frac{54\pi}{5}$

$\therefore \frac{5}{\pi}\sum_{i=1}^{\infty} A_i \Rightarrow \frac{5}{\pi} \times \frac{54\pi}{5} = 54$